Appendix A: $\mu(T)$ for $kT \ll E_F$

$$N = \underbrace{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{A} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon = A\int_0^\infty \frac{\varepsilon^{1/2}\,d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Recall: $T=0$ case, $N = A\int_0^{E_F} \varepsilon^{1/2}\,d\varepsilon = \frac{2}{3}A\,E_F^{3/2}$

"same N"

$$\boxed{\textbf{Sommerfeld Formula}\quad \int_0^\infty \frac{f(\varepsilon)}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon \approx \int_0^\mu f(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}(kT)^2 \left.\frac{df(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\mu} + \cdots}$$

← some function of ε

e.g. in the "N-equation", $f(\varepsilon) = A\,\varepsilon^{1/2}$

$$\underset{\underset{\text{(T=0 result)}}{\uparrow}}{N = } \frac{2}{3} A E_F^{3/2} = \int_0^\infty A \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon \approx \int_0^{\mu} A\varepsilon^{1/2} d\varepsilon + \frac{\pi^2}{6}(kT)^2 A \left(\frac{d\varepsilon^{1/2}}{d\varepsilon}\right)_{\varepsilon=\mu}$$

μ ← unknown

μ ← unknown

$$= \frac{2}{3} A \mu^{3/2} + A \frac{\pi^2}{12} \frac{(kT)^2}{\mu^{1/2}}$$

(math. done!)

$$= \frac{2}{3} A \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu}\right)^2 \right]$$

(aim is to put μ as subject)

$$E_F^{3/2} = \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu}\right)^2 \right]$$

ignore $\left(\frac{kT}{\mu}\right)^4 \cdots$

$$\mu(T) = E_F \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu}\right)^2 \right]^{-2/3} \approx E_F \left[1 - \frac{2}{3} \frac{\pi^2}{8} \left(\frac{kT}{\mu}\right)^2 + \cdots \right]$$

$$\approx E_F \left[1 - \underbrace{\frac{\pi^2}{12} \left(\frac{kT}{E_F}\right)^2}_{\text{leading correction term}} \right]$$

$$\therefore \mu(T) = E_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{E_F}\right)^2 \right]$$

or $$\frac{\mu(T)}{E_F} = 1 - \frac{\pi^2}{12} \left(\frac{kT}{E_F}\right)^2$$

3D Ideal Fermi Gas

Appendix B : $E(T)$ for $kT \ll E_F$

$$E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1}\, d\varepsilon = A \int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon$$

$$\approx A\left[\int_0^\mu \varepsilon^{3/2} d\varepsilon + \frac{\pi^2}{6}(kT)^2\left(\frac{d}{d\varepsilon}\varepsilon^{3/2}\right)_{\varepsilon=\mu}\right]$$ (Sommerfeld Formula)

$$= A\left[\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(kT)^2\mu^{1/2}\right] = A\frac{2}{5}\mu^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]$$

[Now, $\mu(T)$ is known, $\frac{\mu(T)}{E_F} = 1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2$ (see Appendix A)]

$$= \underbrace{\frac{2}{5}A E_F^{5/2}}_{} \underbrace{\left(\frac{\mu}{E_F}\right)}_{}{}^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]$$

$$\cong E(T=0)\cdot\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right]^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{kT}{E_F}\right)^2\right]$$

$$E(T) \cong E(T=0)\left[1-\frac{5}{2}\frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right]\left[1+5\frac{\pi^2}{8}\left(\frac{kT}{E_F}\right)^2\right]$$

$$\approx E(T=0)\left[1+\pi^2\left(\frac{5}{8}-\frac{5}{24}\right)\left(\frac{kT}{E_F}\right)^2 + \text{ignored}\right]$$

$$= E(T=0)\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\right]$$

(done! shift is tiny, $\left(\frac{kT}{E_F}\right)^2$ vs 1)

$$= E(T=0) + \frac{5\pi^2}{12}\cdot\left(\frac{2}{5}g(E_F)\cdot E_F^2\right)\cdot\frac{(kT)^2}{E_F^2}$$

$$= E(T=0) + \frac{\pi^2}{6}g(E_F)\cdot(kT)^2$$

(result)

$$C = \frac{\partial E}{\partial T} = \frac{\pi^2}{3}k^2 g(E_F)\cdot T = \gamma T$$